JOURNAL OF THE EXPERIMENTAL ANALYSIS OF BEHAVIOR 1992, 57, 377-391 NUMBER 3 (MAY)

APPLYING LINEAR SYSTEMS ANALYSIS TO DYNAMIC BEHAVIOR

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In this paper we present an abbreviated discussion of the linear systems analysis in the time domain. We then consider the qualitative character of the behavioral dynamics predicted using the linear form of the analysis. The analysis is then extended to a second-order form. We illustrate some relevant new features introduced by the second-order form with a special case example.

Key words: behavior dynamics, linear systems, nonlinear systems, schedule transitions

Linear systems analysis has proven useful in describing the behavior seen in a number of situations, principally the steady-state behavior supported by either variable-interval (VI) schedules (McDowell, 1980; McDowell, Bass, & Kessel, 1983; McDowell & Kessel, 1979), or variable-ratio (VR) schedules (McDowell & Wixted, 1986, 1988). Schedules with time-dependent properties represent a new type of test for the analysis. The goal of this paper is to give the qualitative features predicted for the operant behavior of an organism to a schedule with a step transition in its reinforcement rate.

In our past calculations, we have employed a frequency-domain analysis. References to the frequency domain mean the discussion concerns properties of behavior, such as rate, that extend over time. Although the frequency domain is convenient for problems involving average rates, the underlying basis for the analysis can be somewhat less than obvious. We have recently developed an equivalent formulation of the analysis (McDowell, Bass, & Kessel, in press) that uses the time domain. When one refers to the time domain, one is talking about behavior as it occurs in real time. In addition to a clearer basis for past results, the new formulation provides an extension to a more general analysis. As will be shown in the latter half of this paper, this more general analysis may prove useful in understanding behavioral dynamics.

We begin the paper with a development for the time-domain version of our linear analysis. The resulting general form is an integral equation. The structure of the equation and the kernel appearing in the integrand are then discussed. With this framework established, we consider the linear dynamics expected from a simple transition in reinforcement rate. Next, we return to the question of how behavior depends on the reinforcement schedule and consider an extension of the analysis to include quadratic or second-order effects. Finally, we show some of the new dynamics possible with a second-order form, again with reference to a simple transition in reinforcement rate.

LINEAR SYSTEMS ANALYSIS IN THE TIME DOMAIN

There is a convenient development for our application of the linear systems theory that can be expressed solely in the time domain.

This paper was presented at the Jacksonville Conference on Behavior Dynamics held at Jacksonville State University, Jacksonville, Alabama, June 1990. J. J McDowell received support for this work from the National Science Foundation (BNS 8908921). Robert Kessel would like to acknowledge support for this work by Mullard Space Science Laboratory, which is supported by a grant from the Science and Engineering Research Council. We would like to thank the organizers of the conference, Jack Marr and Bill Palya, for all their efforts. In addition to a stimulating intellectual atmosphere, they set a wonderfully high standard for food at a conference. Also we would like to thank Tony Nevin and John Staddon for, respectively, chairing and serving on the panel discussion of our paper. Additionally, we would like to thank Billy Baum, John Gibbon, Peter Killeen, Jack Marr, Joel Meyerson, Tony Nevin, Glen Sizemore, and especially John Staddon for the interesting questions they raised during the panel discusison. We have included some material from the panel in this paper. Finally, we thank Ramona Kessel, Mike McKinley, Mark Pevey and Lisa Shaw for their helpful comments on a previous draft of this paper. Ronald Bass's address is Ronald Bass & Associates, P.O. Box 6907, Colorado Springs, Colorado 80934-6907. Robert Kessel's current address is Department of Psychology, Emory University, Atlanta, Georgia 30322. Requests for reprints should be sent to J. J McDowell, Department of Psychology, Emory University, Atlanta, Georgia 30322.

We begin this development with the assertion that a specific behavior at any particular moment is a linear function of past reinforcers contingent on that behavior. This assumption of linearity will be relaxed later when considering the second-order formulation. To specify completely the form of our analysis will require addressing two more aspects of simple schedule-controlled behavior. We must allow for a description of how individual reinforcers delivered at different times prior to the behavior affect that behavior, *and* we must allow for variations among organisms and species in this description. Because the second and third assumptions only require the analysis to permit an expected type of variability, they are rather softer or more subtle assumptions than the assumption of linearity. Rather than jumping directly to a mathematical formulation of the analysis, we will discuss these three underlying assumptions in more detail.

The basic premise for the time-domain form of our analysis is the existence of a linear relation between behavior and the reinforcement schedule. The usual experimental paradigm for a behavior analyst is one in which a response produces some consequence according to a particular schedule of reinforcement and, after some period of time, the organism shows some characteristic pattern of operant behavior. To make a succinct statement of our linear premise within such an experimental setting, a few words about what is meant, in general, by a linear model may prove helpful. Linear models deal with situations in which a specific degree of freedom is altered by a probe under the experimenter's control. The basic idea of the experiment is that the probe alters the system somewhat, and the resulting change in the system's degree of freedom has a linear dependence on the probe's intensity. For example, one applies a known force to a spring and then relates the resulting elongation to the applied force with a constant of proportionality. A condition that is usually concomitant is that the probe's intensity is modest. For most systems in nature, if you drive them hard enough the linearity will fail in some way (e.g., when the stress on a spring exceeds the elastic limit and the spring undergoes plastic deformation, or when sound intensity shatters a crystal, etc.). Despite this type of restriction, a linear treatment of a system will often yield a tractable mathematical problem that can be studied using an extensive set of methods. So unless one has a good a priori reason to begin with a nonlinear model, a linear treatment is the standard opening of choice. Returning to the analysis, we can now connect a general linear model to the familar operant conditioning experiment. The probe used is the reinforcement schedule from its start at time zero to some given time *t*, and the degree of freedom is the organism's operant behavior at that given time *t*. The development to follow will yield a general form for an organism's operant behavior at the end of a span of time during which a reinforcement schedule has been in effect.

A second assumption basic to our analysis is that reinforcers occurring at different times prior to the behavior affect the behavior with different magnitudes. As an example, a reinforcer delivered 10 s ago is assumed to have more impact on present behavior than a reinforcer delivered 2 weeks ago. A form of our second assumption is a standard working premise in much experimental work. If responding is allowed to stabilize under one schedule and then to stabilize again under a second schedule, it would not make much sense to compare the performance during the two conditions unless one is confident that the steady-state performance in the second condition is independent of the first condition. The required independence is usually demonstrated by showing that the behavior supported by the two conditions does not show ordering effects. In our analysis, we are extending the notion to include the possibility that the effect of a reinforcer fades to zero by some characteristic process. However, because a detailed knowledge of how a reinforcer's effect fades with time does not presently exist, we will set up the analysis with sufficient generality to allow consideration of any possible process. Hence, the second assumption is primarily an assumption of existence.

A final assumption allows for the expected variability among organisms by asserting that the time scales relevant to the behavior can vary from organism to organism. In its simplest form, this third assumption states that the characteristic process by which the effect of a reinforcer fades to zero can be different for different organisms. As will be seen, the

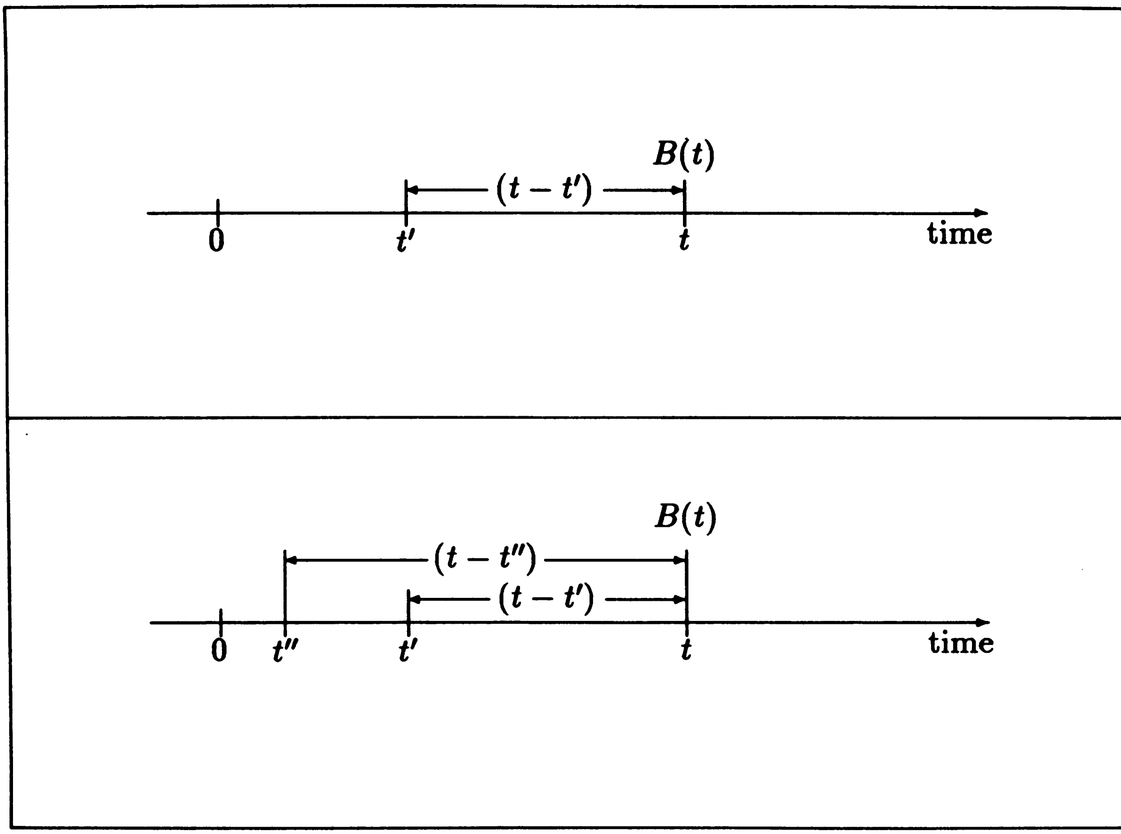

Fig. 1. The top panel shows the relation of $B(t)$ and the time of a single reinforcer delivered at time t' within the interval $0 \rightarrow t$. The bottom panel shows the relation of $B(t)$ and the times of two reinforcers delivered at times t' and t'' within the interval $0 \rightarrow t$.

first two assumptions will determine the general form for a linear dependence of behavior upon the reinforcement schedule. The effect of the varying time scales of the third assumption, on the other hand, is isolated to changes in time scale and form for one mathematical function used within the analysis. These changes in form reflect the differences among organisms. At first pass the third assumption may seem merely a restatement of the second. Its emphasis, however, is rather different. To satisfy the third assumption, the analysis must allow consideration of any time scale.

Using these three assumptions we can develop a general expression for behavior. Our development will move from the linear formulation for an extremely simple reinforcement schedule to that used with increasingly general schedules.

One Reinforcer

Consider a single reinforcer in the interval $0 \rightarrow t$ and then ask what behavior a linear model predicts at the end of the interval. The arrangement of times is shown in the top panel of Figure 1. If the delivery time t' of the reinforcer is fixed, then the expression for the behavior with the posited linear dependence upon reinforcement is

$$B(t) = GR(t'), \tag{1}$$

where G is some constant characteristic of the organism. By using this form we are tacitly acknowledging that the behavior is connected only to the single reinforcer at t', because we have ignored the rest of the interval $0 \rightarrow t$. For the purposes of this paper, it is sufficient to note that $R(t)$ and $B(t)$ describe the measurable characteristics of reinforcement and behavior as a function of time. Beyond requiring that functions exist that both give a faithful representation of the experimental situation and are piecewise continuous, the functional form will not affect the results. For a detailed discussion of a form useful as a representation of $R(t)$ and $B(t)$, the interested reader can consult our earlier work (McDowell et al., 1983; McDowell & Kessel, 1979).

Our first generalization is to allow the delivery time for the single reinforcer to be anywhere within the interval $0 \rightarrow t$. The second assumption, that the length of time elapsed since the reinforcer's delivery affects its impact on behavior, now comes into play. As a result, the expression for a linear dependence of behavior upon the reinforcer will become slightly more complex. As shown in Figure 1, the elapsed time is given by the difference $t - t'$. A large value for this time difference would place the reinforcer's delivery far into the past; a small time difference indicates a recent reinforcer. To satisfy our second assumption, the analysis must differentiate between these various possible time differences. The second assumption is expressed mathematically by introducing a dependence on the time difference $t - t'$. The linear expression for the behavior is given by

$$B(t) = G(t - t')R(t'). \tag{2}$$

The form of Equation 2 is nearly the same as that obtained when t' is fixed. The constant of proportionality G becomes a weighting function dependent on the time difference $t - t'$ and denoted by $G(t - t')$. The function $G(t - t')$ describes how significant an individual reinforcer is to the organism's behavior at time t when delivered at different times prior to the behavior. An extended discussion of this weighting function, $G(t - t')$, will be presented below. It is important to note that Equation 2 is still a linear relationship between reinforcement and responding, irrespective of the form used for $G(t - t')$.

Two Reinforcers

Next consider a schedule that delivers two reinforcers within the interval $0 \rightarrow t$. One of the reinforcers is delivered at the time t', the second reinforcer at the prior time t''. The arrangement of the two reinforcers is shown in the bottom panel of Figure 1. Each reinforcer makes some contribution to the behavior at time t. The behavior due to the reinforcer at t' alone would be

$$B_{t'}(t) = G(t - t')R(t'). \tag{3}$$

Similarly, the other reinforcer in this example by itself results in the behavior

$$B_{t''}(t) = G(t - t'')R(t''). \tag{4}$$

Using the property of linear systems called *linear superposition*, the behavior due to both reinforcers is

$$B(t) = G(t - t')R(t') + G(t - t'')R(t''). \tag{5}$$

If the effects of the two reinforcers are not independent, so that one is unable simply to add their contributions as in Equation 5, then the underlying system is nonlinear. This is not an additional assumption required by the analysis, but something that follows in any linear system. In fact, an equation like Equation 5 is commonly used as the definition for a linear system.

General Linear Form

A general form for the linear dependence of the behavior at time t is found by continuing the superposition used to arrive at Equation 5. For a set of reinforcers delivered at the set of times $t_1, t_2, t_3, \ldots, t_N$, the generalization of Equation 5 is

$$B(t) = G(t - t_1)R(t_1) + G(t - t_2)R(t_2) + \cdots + G(t - t_N)R(t_N), \tag{6}$$

or using a summation to make the expression a bit neater,

$$B(t) = \sum_{n=1}^{N} G(t - t_n)R(t_n). \tag{7}$$

Our usual representations for $R(t)$ and $B(t)$ (McDowell et al., 1983; McDowell & Kessel, 1979) involve piecewise continuous functions. Hence, we need to convert the sum to an integration. The resulting expression for the behavior is

$$B(t) = \int_0^t G(t - t')R(t')dt'. \tag{8}$$

In Equation 8 each segment of $R(t)$ within the interval $0 \rightarrow t$ will contribute to $B(t)$. Effectively, Equation 8 is a general linear mapping from the reinforcement schedule over an interval of time to the operant behavior at the end of the time interval.

We used the interval $0 \rightarrow t$ throughout the development leading to Equation 8 as the period of time over which a reinforcement schedule was in effect. There is nothing fundamental to this specific span of time. One can generalize slightly and frame the analysis using t_o, where $t_o < t$, as the starting time. Switching the interval for the reinforcement schedule under consideration to $t_o \rightarrow t$ simply changes the lower integration limit of Equation 8 as

$$B(t) = \int_{t_o}^{t} G(t - t')R(t')dt'. \qquad (9)$$

The generalization expressed in Equation 9 is useful for cases in which one wishes to assign time zero some other significance. The reader should note that Equation 9 simply allows one to translate the analysis leading to Equation 8 to other intervals of time that may be of interest. It is just a matter of taste whether one refers to Equation 8 or Equation 9 as the basis of the linear analysis in the time domain. In the present paper we will use Equation 9 to handle negative times.

As we noted at the outset, one can reach Equation 8 without reference to the detailed forms for $R(t)$ and $B(t)$. The development for Equation 8 rests on the assumption of linearity and the assumption that reinforcers delivered at different times can have varying impacts on an organism's behavior. Of course, Equation 8 would be of very limited use if one were unable to find functions that faithfully represent the character of the reinforcement schedule and the supported behavior. Consequently, Equation 8, or a related form, can be thought of as the framework for our analysis. Selection of an appropriate representation for $R(t)$ and $B(t)$ an the work required to calculate specific results (e.g., a rate equation) are the other parts of our analysis. Working in the frequency domain to compute rate equations, as we have done in the past, will emphasize this latter part of the analysis.

It is important to note that our earlier formulation of the analysis can be readily recovered from Equation 8. All that is involved is the transformation of Equation 8 to its equivalent frequency-domain form using Laplace transforms. Hence, if one wished, one could go back and rederive our past results.

Kernels and the Structure of Equation 8

In equations with the structure of Equation 8, the weighting function, $G(t - t')$, is called a kernel. In general, a kernel's action is the transformation of one function into a second function with the aid of an integration. (The formal name for the complete integral relation such as Equation 8 is a functional. One could describe the relationship between $B(t)$ and $R(t)$ by saying that $B(t)$ is a linear functional of $R(t)$.) Consider the elements that make up Equation 8. The description of the reinforcement schedule is isolated in the function $R(t)$. Similarly, $B(t)$ isolates the description of the behavior to a second function. The idea that we can use such representations has been around since the earliest form of our analysis (McDowell & Kessel, 1979). The third element of Equation 8 is the kernel $G(t - t')$. Now, recall that $G(t - t')$ first arose in Equation 2 as the proportionality constant relating the input (i.e., the reinforcement schedule) and the output (i.e., the measured operant behavior). As the kernel in Equation 8, $G(t - t')$ is the connection for a general linear transformation of $R(t)$ into $B(t)$. Hence, in our analysis, $G(t - t')$ contains the description of the organism under study. Additionally, the analysis is framed so that the kernel, $G(t - t')$, is independent of the reinforcement schedule, as can be seen by inspection of Equation 8.

It turns out that one need not know $G(t - t')$ exactly to get some results, as we have shown previously (McDowell et al., 1983; McDowell & Kessel, 1979; McDowell & Wixted, 1986, 1988). In this paper we will be using a specific form for the kernel to illustrate some qualitative features of the behavioral dynamics predicted by our analysis. Some further discussion of Equation 8's kernel is needed to provide a context for the kernel form we are going to employ.

Although the form of $G(t - t')$ is presently unknown, there are some conditions that $G(t - t')$ should satisfy. These are: (a) A reinforcer delivered a very long time ago is unimportant, so $\lim_{t-t' \rightarrow \infty} G(t - t') = 0$. (b) Also because an organism cannot respond instantly to a reinforcer, $\lim_{t-t' \rightarrow 0} G(t - t') = 0$. (c) Because organisms do, in fact, respond on schedules of reinforcement, the kernel has to be nonzero

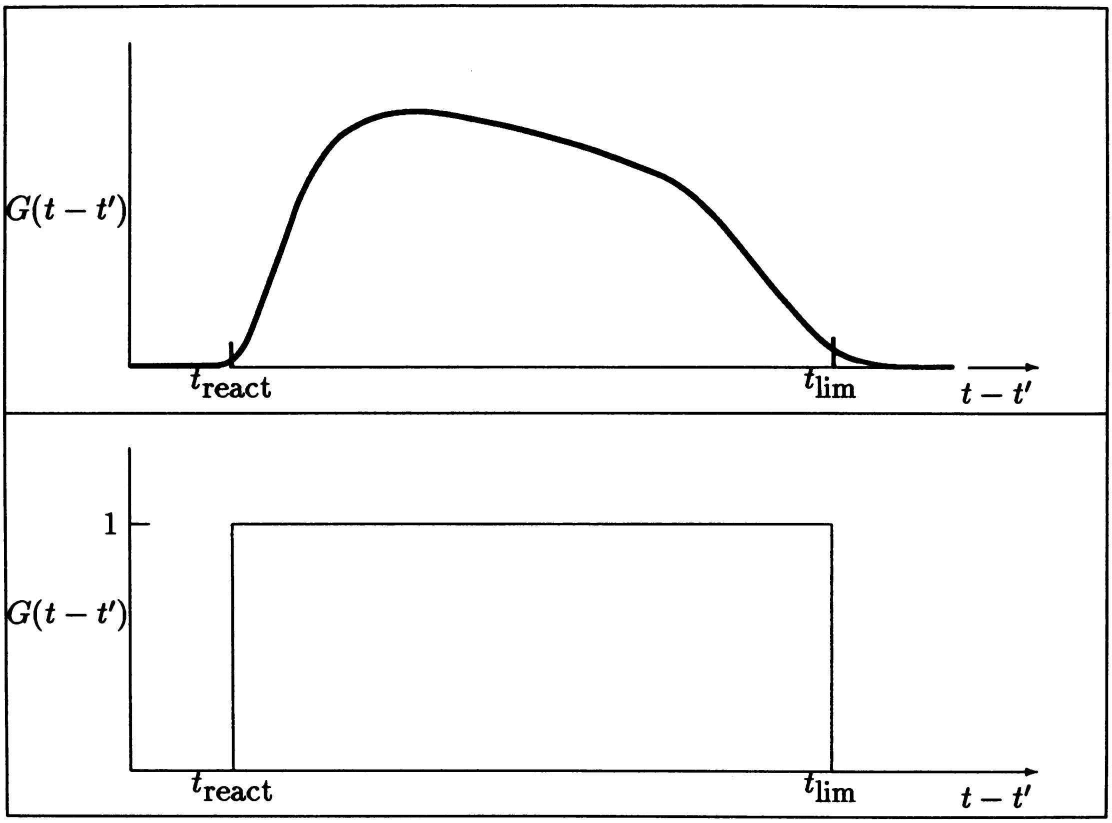

Fig. 2. The top panel shows a possible form of the kernel in Equation 8. The bottom panel shows the form of a rectangular kernel that can be used as an approximation of the more general form shown in the top panel.

between these two limits. A possible form for the kernel of Equation 8 that satisfies these conditions is shown in the top panel of Figure 2.

In Figure 2 the time difference $t - t'$ is a time into the past. Small values of $t - t'$ are very recent times; for example, t_{react} is the shortest time before a reinforcer can have an effect on behavior. As shown in Figure 2, $G(t - t')$ is small until after t_{react} has elapsed. Similarly, large $t - t'$ means times long ago, and, for time differences greater than some limiting time t_{lim}, $G(t - t')$ goes to zero. This drop of $G(t - t')$ to zero as the time difference between t and t' increases corresponds to a reinforcer's fade to insignificance as the elapsed time between its delivery and present behavior exceeds some time characteristic of the organism. Hence, as an order of magnitude estimate for t_{lim}, one could use the period of conditioning needed for an organism's steady-state responding to be established. For a typical schedule and a conservative stability criterion, t_{lim} is about 25 sessions. Between these two limits the kernel shown in the top panel of Figure 2 has the finite values needed so that when used with Equation 8 the reinforcement schedule $R(t)$ will generate some operant behavior $B(t)$. There is one further point that should be made about the kernel shown in the top panel of Figure 2. The analysis does not require one universal form. The opposite is likely to be the case. For example, it is reasonable to expect t_{lim} to be considerably longer in humans than in pigeons. Allowing for variations in $G(t - t')$ among different organisms (within as well as across species) is a reflection of the third assumption in our analysis.

In the long term, if the analysis is to be generally useful, one will have to be able to measure $G(t - t')$ on one schedule and then predict the behavior seen with other schedules. This raises a final point on the kernels. There is nothing mysterious about these kernels. They can be determined from sufficiently detailed experimental data. For example, one could work from the local structure data such as that of Palya (1992). One would use such a data set to construct the frequency-domain forms of the reinforcement schedule and operant behavior via either the Laplace or Fourier transforms. The frequency domain form of the kernel then follows as

$$G(s) = \frac{B(s)}{R(s)}. \quad (10)$$

$G(t - t')$ can then be recovered by applying the inverse transform to $G(s)$. The measurement of a kernel and the subsequent demonstration of the kernel's invariance under change of the reinforcement schedule is a critical test of our analysis. Failure of the kernel's invariance would suggest serious flaws in the analysis.

SOME LINEAR DYNAMICS

To illustrate the dynamics predicted by Equation 8 we need an analytic form for the kernel. Although one could find a form that fits the kernel shown in the top panel of Figure 2, a simpler kernel will be sufficient for our present purpose. Perhaps the simplest model that satisfies the conditions we have set uses a rectangular kernel. The expression for a rectangular kernel is

$$G_r(t - t') = \begin{cases} 1 & t_{\text{react}} < t - t' < t_{\text{lim}} \\ 0 & \text{elsewhere} \end{cases}. \quad (11)$$

A plot of Equation 11 is shown in the bottom panel of Figure 2. If $t_{\text{react}} \ll t_{\text{lim}}$, then the rectangular kernel may be simplified further by using the reasonable approximation for the rectangular kernel of

$$G_r(t - t') \simeq \begin{cases} 1 & 0 < t - t' < t_{\text{lim}} \\ 0 & \text{elsewhere} \end{cases}. \quad (12)$$

Because our present purpose is only to find the rough qualitative dynamics expected with a linear analysis, Equation 12 will be employed.

The kernel given in Equation 12 is related to the "time windows" ideas of Staddon (1988) and Killeen (1981), with t_{lim} filling the role of the length of the window. For example, by changing from 1 to $1/J$, where J is Killeen's window length parameter, in the upper condition of Equation 12 and converting Equation 8 to the appropriate discrete summation, one can recover Killeen's moving average. Similarly, Killeen's exponentially weighted moving average can be generated with an exponentially decaying kernel and discrete summation. In both these other analyses the underlying structure, the endpoint behavior dependent on the preceding reinforcement, is similar to our own. However, a detailed comparison of our analysis and other analyses, although interesting, would take us far from the topic of dynamics.

At this point we can now find some qualitative properties of the dynamics predicted by the analysis. The problem we consider is: Using the simplified rectangular kernel of Equation 12, what kind of dynamics are obtained? We will further restrict the problem to one of the simplest situations for which dynamics has relevance: the case of a step transition in the reinforcement schedule. To provide a well-defined special case, consider a transition at the time $t = 0$ between a VI 80-s schedule of reinforcement and a VI 20-s schedule. The transition is unsignaled. The transition between two reinforcement schedules is shown schematically in Figure 3. This situation, where the times before the transition correspond to $t < 0$, will require the use of Equation 9. In this example the VI 80-s schedule has been running since $t \to -\infty$ and the VI 20-s schedule then continues in effect until $t \to +\infty$. Under these circumstances the question of dynamics becomes a question of how the behavior changes after the step transition in reinforcement rate at $t = 0$.

For t_1, which occurs before the transition, the behavior is given by substituting Equation 12 into Equation 9, which yields

$$B(t_1) = \int_{t_1 - t_{\text{lim}}}^{t_1} R_{\text{VI80}}(t')dt'. \quad (13)$$

In Equation 13 the kernel's only effect is to set the limits of integration, because its value is either 1 or 0. The region of time that contributes to the integration in Equation 13 is shown in the top panel of Figure 3. This should be "steady-state" VI 80-s behavior because the contributions to the behavior come solely from the VI 80-s schedule.

The next two times, t_2 and t_3, are transition cases. The expressions of the behavior are

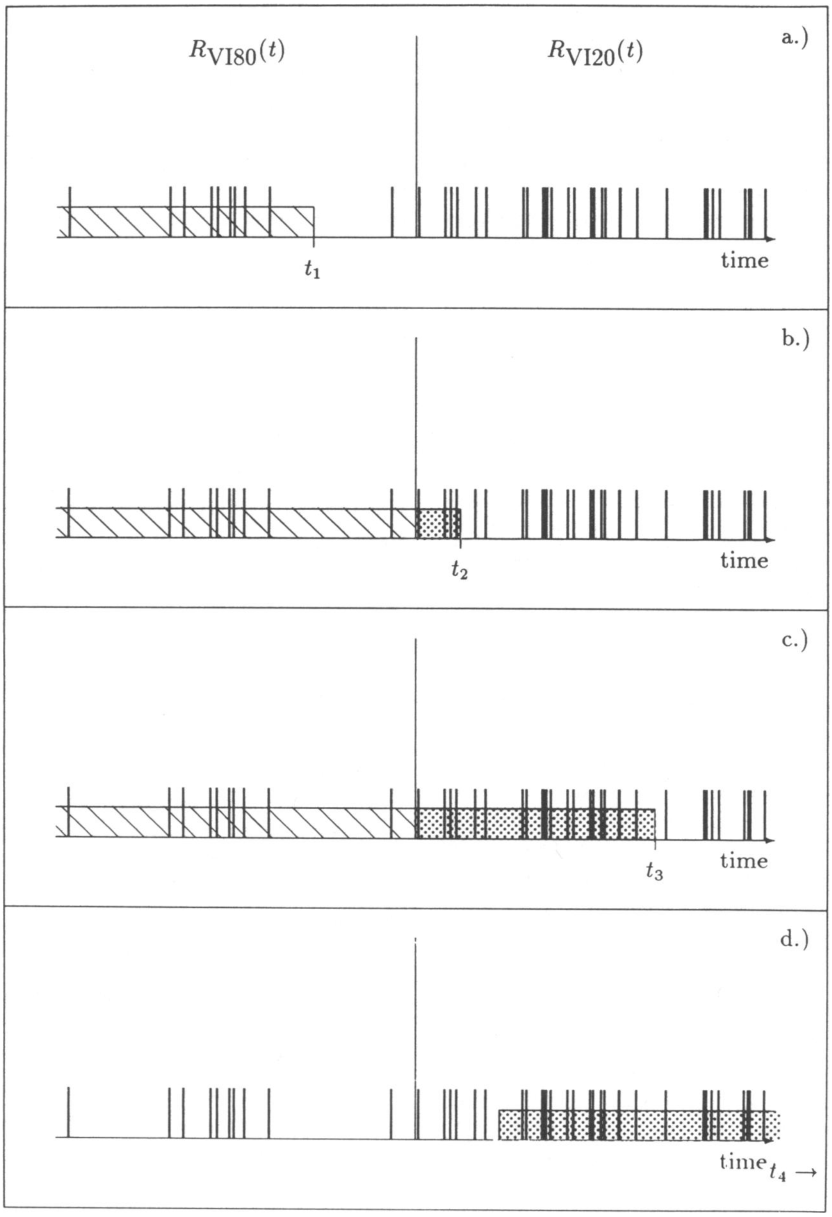

Fig. 3. A step transition in the reinforcement rate of a VI schedule and the regions of time contributing to the integrals of Equations 13, 14, 15, and 16. In each of the four panels, the vertical pips are the locations of the reinforcers. To be completely rigorous one would consider a statistical ensemble of all distributions of reinforcers that satisfy both the VI 80-s schedule for $t < 0$ and the VI 20-s schedule for $t > 0$. The times t_1, t_2, t_3, and t_4 used in computation of the resulting transition in behavior are shown along the time axis of the relevant panel. In (a) only the VI 80-s schedule is contributing. The region of the integration is shown as the crosshatched band beginning at t_1 and extending off the plot to the left. This end of the region is for small $t_1 - t'$ values. The other end of the integration region is off the left edge of the plot where $t_1 - t' = t_{\text{lim}}$. In (b) the VI 20-s schedule begins to make a contribution to the integral. The VI 20-s schedule's contribution comes from the small region between $t = 0$ and $t = t_2$, which is shown as the shaded band. The contribution from the VI 80-s schedule is the crosshatched band running from $t = 0$ off to the left of the plot. In (c) the situation is similar to (b), but as $t_3 > t_2$ the VI 20-s schedule makes a larger contribution to the integral, and as shown on the plot, the shaded band increases in length. Finally in (d) the situation when $t_4 > t_{\text{lim}}$ is shown. The point t_4 is somewhere off the plot to the right. The shaded band of the VI 20-s schedule's contribution seen on the plot is the end of the integration interval where $t_4 - t'$ reaches t_{lim}.

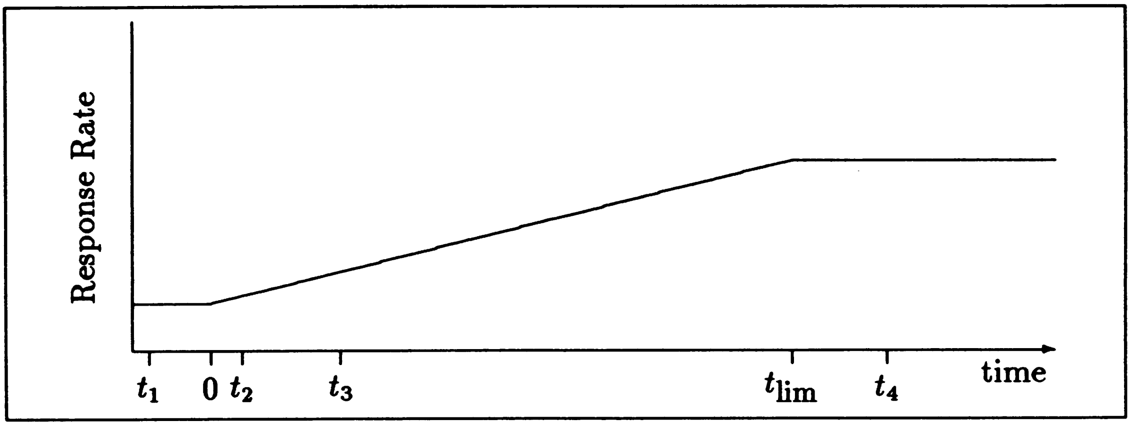

Fig. 4. Response rate as a function of time near a transition in reinforcement rate with a rectangular kernel.

$$B(t_2) = \int_{t_2-t_{\text{lim}}}^{0} R_{\text{VI80}}(t')dt' + \int_{0}^{t_2} R_{\text{VI20}}(t')dt' \quad (14)$$

and

$$B(t_3) = \int_{t_3-t_{\text{lim}}}^{0} R_{\text{VI80}}(t')dt' + \int_{0}^{t_3} R_{\text{VI20}}(t')dt'. \quad (15)$$

In Equations 14 and 15, there are contributions from both schedules. The regions of time where the VI 80-s and VI 20-s schedules will contribute to the behavior at times t_2 and t_3 are shown in the middle two panels of Figure 3. Because the VI 20-s schedule supports a higher steady-state response rate, there should be a higher response rate at t_3 because Equation 15 has a larger contribution from the VI 20-s schedule.

Finally, at t_4 where $t_4 > t_{\text{lim}}$, a new steady state, the VI 20-s steady state, is obtained. The expression for this new steady state is given by

$$B(t_4) = \int_{t_4-t_{\text{lim}}}^{t_4} R_{\text{VI20}}(t')dt'. \quad (16)$$

Steady-state VI 20-s responding will occur at t_4 because the span of time allowed by substituting Equation 12 into Equation 9 comes solely from the VI 20-s schedule. One end of this span of time is shown in the bottom panel of Figure 3.

The expected plot of the response rate is shown in Figure 4. The dynamics are slow because the time of the transition is set by t_{lim}. The slope of the ramp will go as $1/t_{\text{lim}}$. Equations 13, 14, 15, and 16 describe steps along a quite simple transition process. The contributions from the VI 80-s schedule (Equation 13 and the first integral in Equations 14 and 15) slowly die out and are replaced by contributions from the VI 20-s schedule (the second integral in Equations 14 and 15 and Equation 16). Note that because it is a parameter of the kernel, t_{lim}, that sets the scale for the transition in behavior, one would obtain the same qualitative result (i.e., a linear ramp) if any pair of schedules were considered.

We used Equation 12 for calculational ease. How does the form of the organism's transition in behavior change if we use a more realistic kernel? The question can be answered in the following manner. First, it follows from linear superposition that one can always express the behavior at a given time t after the VI 20-s schedule comes into effect as the sum of two integrals, such as in Equations 14 or 15. For the case of a general linear kernel, the behavior, at a time $t > 0$, with a transition from VI 80 s to a VI 20 s at time zero, is given by

$$B(t) = \int_{-\infty}^{0} G(t-t')R_{\text{VI80}}(t')dt' + \int_{0}^{t} G(t-t')R_{\text{VI20}}(t')dt'. \quad (17)$$

Second, as we have already mentioned, the idea that steady-state operant behavior is established by a reinforcement schedule requires that, within our analysis, after a span of time t_{lim} has elapsed, a prior reinforcer will have no effect on present behavior. Hence, for the case when $t > t_{\text{lim}}$, the first integral on the right hand side of Equation 17 will be driven to zero by $G(t - t')$'s approach of zero for $t - t' > t_{\text{lim}}$. Additionally, the limited domain of nonzero values for $G(t - t')$ will truncate the integration interval for the second integral on the right side of Equation 17. If we ignore the precise details of how $G(t - t') \to 0$ and simply use t_{lim} as a hard cutoff, the second integral on the right side of Equation 17 for $t > t_{\text{lim}}$ can be approximated as

$$B(t) \approx \int_{t-t_{\text{lim}}}^{t} G(t - t')R_{\text{VI20}}(t')dt'. \quad (18)$$

Equation 18 is an expression for the steady-state VI 20-s behavior seen for all times after t_{lim}. One can simplify further by noting that a VI 20-s schedule's properties do not depend on time[1] so that any interval of length t_{lim} can be used, and hence, the steady-state VI 20-s behavior is

$$B_{\text{VI20}}(t) \approx \int_{0}^{t_{\text{lim}}} G(t_{\text{lim}} - t')R_{\text{VI20}}(t')dt', \quad (19)$$

where $t > t_{\text{lim}}$. By a similar argument one can show that the expression for the behavior just before the transition in reinforcement schedule, $B(0^-)$, is a suitable approximation for the steady-state VI 80-s behavior. The appropriate expression is

$$B_{\text{VI80}}(t) \approx \int_{-t_{\text{lim}}}^{0} G(-t')R_{\text{VI80}}(t')dt', \quad (20)$$

for all $t < 0$. In summary, Equations 19 and 20 provide expressions for steady-state behavior before and after the transition. Additionally, Equation 17 describes the transitions, effectively the dynamics, that follow from a linear analysis. Finally, the transition region will be $\sim t_{\text{lim}}$ in length independent of the specific form used for $G(t - t')$.

At this point, although one might be willing to accept that Equation 17 is the general expression for an organism's operant behavior when driven by a step transition in reinforcement schedule, one might also find that its application to data may not seem immediately obvious. The computations needed for a quantitative description of behavior within the transition region would take us far beyond the goals of the present paper. However, by considering the ratio of behavior generated by the VI 20-s reinforcement schedule within the transition to steady-state VI 20-s behavior, we can obtain a rough idea for the dependence of the transition's form upon the kernel. The ratio of the second integral in Equation 17 to Equation 19,

$$\frac{B(t)}{B_{\text{VI20}}(t)} = \frac{\int_0^t G(t - t')R_{\text{VI20}}(t')dt'}{\int_0^{t_{\text{lim}}} G(t_{\text{lim}} - t')R_{\text{VI20}}(t')dt'}, \quad (21)$$

will give the proportion of the behavior due to the VI 20-s schedule. Because the VI 20-s schedule's properties do not depend on time, the time evolution of Equation 21 will be a result of $G(t - t')$'s time dependence. Hence the ratio will be approximately

$$\frac{B(t)}{B_{\text{VI20}}(t)} \approx \frac{\int_0^t G(t - t')dt'}{\int_0^{t_{\text{lim}}} G(t_{\text{lim}} - t')dt'}. \quad (22)$$

Assuming that properties of the behavior, such as local rate, also are proportional to Equation 22, one can use this expression to estimate how such properties evolve during the transition for any given kernel. For example, substitution of Equation 12 into Equation 22 yields the linear ramp of Figure 4. Similarly, if one employed an exponential kernel proportional to $e^{-(t-t')/\tau}$, the steady-state VI 20-s behavior will replace the VI 80-s behavior as $\sim e^{-t/\tau}$. By setting τ to $\frac{1}{6} t_{\text{lim}}$, one would predict a more rapid initial change in behavior followed by a slow settling time of about t_{lim} in duration.

Another experimental situation that can test the dynamics predicted by the linear analysis are reinforcement schedules with a periodic variation in their properties. Examples of periodic schedules are Staddon's cyclic-interval (1964) and McDowell and Sulzen's ramp schedules (1981). One probably cannot explain the behavior supported by either of these schedules within a linear formulation. On such

[1] We are making a distinction between the properties of a VI 20-s schedule, such as average rate of reinforcement, that are unchanging with the passage of time and the schedule itself, which clearly does depend on time. Additionally, we are glossing over such details as the ensemble averaging necessary in a proper calculation with a VI schedule.

periodic schedules, which have variations that are fast in comparison to the estimate of $t_{\lim}$ obtained from the time to steady-state behavior, the predicted behavior would be just an averaged rate, rather than the slewed rate of responding that is actually measured. At a minimum, for an organism to sense, and hence respond to, a local change in a reinforcement schedule's properties, it must make correlated reference to two times in the past. The independent references to prior reinforcers inherent in Equation 8 are insufficient to estimate changes in the underlying schedule. The basic point is this: Without using the time interval and ordering of the two references, the organism could not determine a trend. The notion that an organism's current behavior can depend on the correlations within the reinforcement schedule at two prior times foreshadows the contents of the balance of this paper.

QUADRATIC (SECOND-ORDER) ANALYSIS IN THE TIME DOMAIN

Although the linear analysis has had considerable success in describing steady-state (static) behavior, it appears to have some analytic properties that are at odds with what is known about behavioral dynamics. Because Equation 8 is based on linear superposition, the analysis cannot describe behavior that has an explicit dependence upon correlations within the reinforcement schedule. The simplest form of correlation involves the simultaneous consideration of the reinforcement schedule at two times prior to the present behavior. The idea that present behavior depends on the reinforcement schedule at *two* times in the past can be expressed as a quadratic relationship between present behavior and past reinforcement. It turns out that our analysis has a nonlinear extension that, as a special case, can describe a relationship between two-point correlations in the reinforcement schedule and behavior. In the balance of this paper we will explain just the quadratic extension of our analysis and then show how the inclusion of two-point correlations can yield a different response to the simple VI 80-s to VI 20-s transition discussed above.

One method to extend the time-domain version of our linear systems analysis is to add a term that has a quadratic dependence on the reinforcement schedule over the interval $0 \to t$. To obtain the most general quadratic dependence means forming a product of the reinforcement schedule over the interval $0 \to t$ with itself. In practice, a quadratic dependence will mean that for a set of discrete reinforcers, the second-order behavior is given by the square of a sum of terms. To get both an idea of why a quadratic analysis will prove useful and as a suggestion of the route ahead, the reader should consider what the square of a sum of terms involves. Stripped of the mathematical niceties that are needed for computational convenience, including a quadratic dependence for a set of discrete reinforcers within the analysis involves finding the square of the right side of Equation 6. Note that the cross terms from such a binomial expansion will involve reinforcers delivered at two different times. It will be these cross terms that give rise to the two-point correlations. For the more general case of a continuous reinforcement schedule, the second-order form analogous to Equation 8 will come out as a double integral. The new term can also be called *second order* because there will be two factors of the reinforcement schedule involved. We denote the new term as $B_2(t)$, so the behavior is now given as

$$B(t) = B_1(t) + B_2(t), \qquad (23)$$

where $B_1(t)$ is the behavior of Equation 8 alone.

At this point, we should note that we arrived at our expression for $B_2(t)$ from a somewhat different set of considerations. Equation 8 is reminiscent of the first term in a series expansion developed by Kubo to describe irreversible processes (Kubo, 1957; Kubo & Tomita, 1954). Kubo's ideas suggested a method by which the linear analysis for simple-schedule behavior could be extended as a series of increasingly nonlinear terms. Accordingly, there are higher order terms (e.g., cubic, quartic, etc.) that can be added to Equation 23. The more general and formal considerations that lead to the full series expansion are discussed elsewhere (McDowell et al., in press).

We will develop the second-order term $B_2(t)$ by a route roughly analogous to that used in reaching the first-order expression of Equation 8. As in the linear development, the most significant action concerns the construction of the kernel. Because the second-order term speaks to situations in which two prior times have a simultaneous effect on present behavior, the second-order kernel will have two time difference arguments. The reader should pay special attention to the introduction and development of the second-order kernel's arguments.

One Reinforcer in Second Order

For a single discrete reinforcer in the interval $0 \rightarrow t$, the behavior due to just a second-order process is given by

$$B_2(t) = G_2(t - t')[R(t')]^2, \quad (24)$$

where $G_2(t - t')$ is a characteristic of the organism. Because only a single time prior to the present behavior, t', is of relevance, the second-order kernel can only depend on one time difference, $t - t'$, in the single reinforcer situation. As a result, Equation 24 does not suggest any obvious new effects. We included this situation mainly to establish notation. Note that $G_2(t - t')$ will in general be different than $G(t - t')$.

Two or More Reinforcers in Second Order

With two discrete reinforcers in the interval $0 \rightarrow t$, cross terms are possible. The behavior due to just a second-order process is given by

$$\begin{aligned} B_2(t) = {} & G_2(t - t')[R(t')]^2 \\ & + G_2(t - t'')[R(t'')]^2 \\ & + 2\mathbf{G}_2(t - t', t - t'')R(t')R(t''). \end{aligned} \quad (25)$$

$\mathbf{G}_2(t - t', t - t'')$ is a characteristic of the organism that describes the importance to the organism's behavior of pairs of reinforcers. For full generality, we allow for a dependence upon the time difference to both members of the pair. For example, if either member of the reinforcer pair was delivered a very long time prior to the present behavior, one expects $\mathbf{G}_2(t - t', t - t'') \rightarrow 0$. $\mathbf{G}_2(t - t', t - t'')$'s form is new with second order. Notice the difference in arguments between $\mathbf{G}_2(t - t', t - t'')$ and $G_2(t - t')$ that arose in the single reinforcer case. The cross term involving $\mathbf{G}_2(t - t', t - t'')$ is describing an interaction between or the correlated effect of the two reinforcers. One can argue that such interactions exist based on the retroactive and proactive effects of reinforcers measured by Catania, Sagvolden, and Keller (1988).

Although Equation 25 gives all the behaviorally salient aspects for a quadratic dependence upon two reinforcers, it is somewhat unwieldy. With a suitable redefinition, one can get a common form for both $\mathbf{G}_2(t - t', t - t'')$ and $G_2(t - t')$. There are two mathematical points that underlie the redefinition. First, if $t - t'$ and $t' - t''$ are known, then $t - t''$ is also known. Hence one can express $\mathbf{G}_2$ as a function of any two of these three time difference arguments. Second, one can always consider a term like that involving $[R(t')]^2$ as a pair of reinforcers delivered with zero separation. As a result, the two second-order kernels of Equation 25 can use a common form. The second-order behavior due to two reinforcers may be rewritten as

$$\begin{aligned} B_2(t) = {} & G_2(t - t', 0)R(t')R(t') \\ & + G_2(t - t'', 0)R(t'')R(t'') \\ & + G_2(t - t', t' - t'')R(t')R(t''). \end{aligned} \quad (26)$$

Note that the factor of 2 that appears in the third term of Equation 25 can be absorbed into $G_2(t - t', t' - t'')$ as part of the redefinition. A more detailed discussion of the redefinition that steps from Equation 25 to Equation 26 is in McDowell et al. (in press). More important than the redefinition itself is that one can show that the second-order behavior, when driven by any number of reinforcers, will still only have terms of just the two types seen in Equation 26 (McDowell et al., in press). This demonstration proceeds by explicit construction of the second-order term for greater numbers of reinforcers. It is an interesting bit of algebra, but one that has little relevance to dynamics. As noted, the important outcome of these considerations is that there are just two types of terms in second order. The only two types of terms possible as second-order contributions are (a) products of the schedule with itself at a single time, such as $G_2(t - t', 0)R(t')R(t')$; or (b) products of the schedule with itself at two different times, such as $G_2(t - t', t' - t'')R(t')R(t'')$.

General Second-Order Form

The general second-order term for any $R(t)$ extends the summation of Equation 26 to all possible pairs of times in the interval $0 \rightarrow t$. Not too surprisingly, such a sum over all pairs of times is given by a double integral. The expression is

$$B_2(t) = \int_0^t dt' \int_0^{t'} G_2(t - t', t' - t'')R(t')R(t'')dt''. \quad (27)$$

The demonstration that Equation 27 is a proper generalization of Equation 26 for a piecewise continuous $R(t)$ is somewhat lengthy. For the present we will confine the discussion to showing that Equation 27 has both types of second-order contribution that became apparent when considering just two reinforcers. In the complete development of Equation 27, technical details, such as avoidance of double counting, become important. A reader interested in the detailed development of Equation 27 should see McDowell et al. (in press).

To see that Equation 27 contains both types of second-order contribution, first consider the integration over dt''. At the upper limit of the integration $t' = t''$, the integrand of Equation 27 has the form

$$G_2(t - t', 0)R(t')R(t'). \tag{28}$$

This is the product of the reinforcement schedule with itself at a single point in time. The product of the reinforcement schedule with itself at two different times comes from the rest of the interval $0 \rightarrow t'$, because $t'' < t'$ and the integrand has the form

$$G_2(t - t', t' - t'')R(t'')R(t'). \tag{29}$$

The complete second-order term results from the integration over dt' that varies t' across the interval $0 \rightarrow t$. Hence, Equation 27 has the only two types of product possible in second order and sums over such contributions from the entire interval $0 \rightarrow t$.

If one wishes to consider all possible pairs of time within the more general interval of time $t_o \rightarrow t$, then Equation 27 becomes

$$B_2(t) = \int_{t_o}^{t} dt' \int_{t_o}^{t'} G_2(t - t', t' - t'')R(t')R(t'')dt''. \tag{30}$$

The relationship between Equation 27 and Equation 30 is the second-order analogy of that between Equation 8 and Equation 9. As in the linear case, this generalization is useful when the situation dictates a meaning for $t = 0$ other than the start of the reinforcement schedule.

A SEPARABLE KERNEL AND SOME SECOND-ORDER DYNAMICS

The interactions or correlations within the reinforcement schedule at different times can allow a description of an organism's sensitivity to transitions in the reinforcement schedule. As with the earlier discussion of linear dynamics, we will employ a special kernel. This special second-order kernel will provide a simple illustration of the role pair interactions or two-point correlations play in second order. Once we have shown how this special second-order kernel isolates a particular reinforcement pairing, we will return to the step transition schedule. The reader should bear in mind that statements about the second-order term based on the one special case kernel are somewhat speculative. In other words, what follows is simply an example of how the second-order term can be used to describe a dynamic effect.

Consider the kernel

$$G_2(t - t', t' - t'') = g_2(t - t')\delta(t' - t'' - \Delta t), \tag{31}$$

where $g_2(t - t')$ is a function that describes the effect on operant behavior at time t of reinforcers separated by the fixed time interval Δt.[2] The δ-function allows us to select only those contributions with a separation in time of Δt. Upon substitution of Equation 31 into Equation 27, the second-order term becomes

$$B_2(t) = \int_0^t R(t' - \Delta t)g_2(t - t')R(t')dt'. \tag{32}$$

In Equation 32 one can see explicitly that this form for the second-order behavior connects or correlates the reinforcement schedule at two times in the past. Such a two-point correlation may prove helpful when using the analysis to describe responding through a transition of the schedule.

As an illustration of how the dynamics change when the effects of a local two-point correlation are included, we will again consider the step transition from a VI 80-s to a VI 20-s schedule. For $t < 0$ the mean inter-

[2] This kernel employs the Dirac δ-function, which is defined by

$$\int_a^b f(x)\delta(x - x_0)dx = f(x_0),$$

where $a \leq x_0 \leq b$. Delta functions are used to select out a single contribution to an integral. In the present case, the δ-function is employed to pick off a single time difference in the second-order term. For a more general explanation on the use of δ-functions, an interested reader should consult a mathematical methods text (Arfkin, 1985; Boas, 1983).

reinforcement interval is 80 s, and for $t > 0$ the mean interreinforcement interval is 20 s. So, we modify Equation 31 to pick off just these two time intervals as

$$G_2(t - t', t' - t'') = -g_2(t - t')\delta(t' - t'' - 80) + g_2(t - t')\delta(t' - t'' - 20). \quad (33)$$

The resulting expression for the $t > 0$ behavior will be

$$B(t) = \int_{-\infty}^{t} G_1(t - t')R(t')dt' - \int_{-\infty}^{0} R(t' - 80)g_2(-t')R(t')dt' + \int_{0}^{t} R(t' - 20)g_2(t - t')R(t')dt'. \quad (34)$$

In Equation 34 the two terms that have been added to the linear (first-order) behavior are sensitive to either the interreinforcement interval typical of a VI 80-s schedule (the second integral) or the interreinforcement interval typical of a VI 20-s schedule (the third integral). Effectively what we have done with the δ-function factors of Equation 33 is to create a specialized sensitivity to the expected two-point correlations present in the schedule before and after the transition. By adjusting the time scale $g_2(t - t')$ relative to $G_1(t - t')$'s time scale, given by t_{lim}, one can get the same steady-state behavior and a more reasonable response to a reinforcement schedule transition. For example, if $g_2(t - t')$ is significantly different from zero only over a much briefer domain of $t - t'$ when compared to $G_1(t - t')$'s nonzero domain, then the effect of the second and third integrals is a quick increase in responding after the transition.

We are not asserting that the higher order terms have done anything at present. Instead the above simply demonstrates that the second-order term has, in principle, the necessary analytic properties to sense the local change in two-point correlation that occurs at a transition. Hence, it is possible that a second-order analysis will yield a better description of behavioral dynamics. Whether we can describe an organism's *general* sensitivity to changes in a reinforcement schedule using a second-order analysis is the subject of current work.

CONCLUSIONS

In this paper we have shown what qualitative character is expected for the behavioral dynamics based on the linear systems analysis. We have used a time-domain development of the analysis to provide a framework for the discussion. The dynamics expected for a linear description using a simple rectangular kernel are slow. In addition, the predicted linear behavior is insensitive to a transition itself in the reinforcement schedule (i.e., the correlational properties of the schedule can play no role in determining the organism's behavior in a linear description). The time-domain development for the analysis has suggested a quadratic extension that may describe other types of dynamic processes. Finally, we have shown an example of the resulting second-order dynamics that will take a schedule's transitions into account.

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Received May 28, 1991
Final acceptance December 6, 1991